Mean Field Behavior during the Big Bang Regime for Coalescing Random Walk

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CRW

Coalescing Random Walk (CRW):

- Initially one walker at each vertex of the graph G
- Each walker performs independent continuous time simple random walk.
- Whenever two walkers meet(collide), they merge into one walker. This walker continues to do (CT)SRW.

Can be extended to general Markov chain with rate $\mathbf{r}$. In this talk we focus SRW so that $r_{x,y} = \mathbf{1}[x \sim y]$ (general graph) or $r_{x,y} = 1/d(x)$ (regular graph). Our result also applies to general symmetric rates.

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Motivation: duality with the voter model.

An example

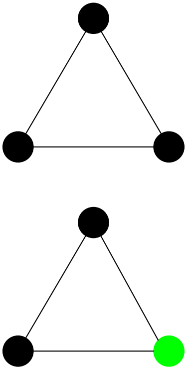

Figure: Black=occupied, Green=vacant.

Observation/Question

Simple facts:

- On a finite graph, the total number of walkers decreases with t and eventually stabilizes at 1.
- On any graph, the probability that a given site is occupied decreases with t.

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Questions

- How long does it take for all walkers to coalesce into one?
- What is the decay rate for the fraction of occupied site/probability of a site being occupied?(main focus)

CRW on the complete graph- Kingman's coalescent

Jump rate across each edge$=1/(n-1)$.
L_t: $\#$ of walkers at time t. τ_{coal}: the (random) coalescence time.
$L_0 = n$ and L_t decrease by 1 at rate $L_t(L_t-1)/(n-1)$. Let e_i be i.i.d.exponential random variables with unit rate, then

$$\tau_{\mathrm{coal}} = \sum_{i=2}^{n} \frac{e_i}{i(i-1)/n}.$$

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Kingman's coalescent: $L_0 = \infty$. $L_t \to L_t - 1$ at the rate of $L_t(L_t-1)/2$.
$P_t = \mathbb{E}(L_t)/n$: expected fraction of occupied sites. Fix t, send $n \to \infty$

$$P_t = 1/(1+t).$$

Spatial structure

Often there is a spatial structure.

- $\mathbb{Z}^d$.
- $\mathbb{T}^d$.
- General vertex transitive graphs.
- Random graphs (e.g., configuration model).

Heuristic argument [van den Berg-Kesten, 2000]

Consider $\mathbb{Z}^d$. $P_t = P_t(o)$: prob. that origin is occupied at time t.
Take $1 \ll \Delta(t) \ll t$.

$$
\begin{aligned}
-\frac{dP_t}{dt} &= \mathbb{P}(o \text{ and } \mathbf{e}_1 \text{ occupied at } t) \\
&\sim \sum_{x,y} \mathbb{P}(x \text{ and } y \text{ occupied at } t - \Delta(t)) \times \\
&\quad \mathbb{P}(x + S_{\Delta(t)} = o, y + S'_{\Delta(t)} = \mathbf{e}_1, x + S_r \neq y + S'_r, \forall r \leq \Delta(t)) \\
&\sim P^2_{t-\Delta(t)} \alpha_{\Delta(t)}.
\end{aligned}
$$

- x and y are the location of the walkers that later come to o and $\mathbf{e}_1$. $S_\cdot, S'_\cdot$: independent random walks starting from o.
- $\alpha_{\Delta(t)}$: the probability that two time-reversed random walk starting from o and $\mathbf{e}_1$ don't collide by time $\Delta(t)$.

Results on $\mathbb{Z}^d$

Assuming $P_t \sim P_{t-\Delta(t)}$ and $\alpha_t \sim \alpha_{t-\Delta(t)}$. The heuristic suggests that $P_t \approx 1/(t\alpha_t)$ for moderately large t. This was known to be true for SRW on $\mathbb{Z}^d, d \geq 2$.

Theorem (Bramson-Griffeath, 1980)

Consider the CRW on $\mathbb{Z}^d$. Ww have, as $t \to \infty$,

$$P_t \sim \begin{cases} \frac{1}{\pi}\frac{\log t}{t} & d = 2 \\ (\gamma_d t)^{-1} & d \geq 3 \end{cases}$$

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By justifying previous heuristic argument, [van der Berg-Kesten, 2000] proved the same result for $d \geq 3$ (their proof also works for general coalescing model which allows for more than one particles per site).

Beyond $\mathbb{Z}^d$-Mean Field Predictions

Set

$$t_{\text{meet}} = \mathbb{E}_{\pi^2}(\tau_{\text{meet}}).$$

Aldous and Fill conjectured that for general transitive graph (transitivity means the graph looks the same from every vertex)

$$\frac{\tau_{\text{coal}}}{t_{\text{meet}}} = \sum_{i=1}^{\infty} \frac{e_i}{i(i-1)/2}.$$

Here $e_i \sim Exp(1)$. Equality holds for complete graphs. Intuition: the time to go from i particles to $i-1$ takes the minimum of $i(i-1)/2$ independent exponentials.

Aldous-Brown approximation

Lemma (Aldou-Brown, 1992)

For an irreducible reversible Markov chain on a finite state V with stationary distribution π and $A \subset V$, if we denote the hitting time of A by T_A and its density function w.r.t. the stationary chain by f_{T_A}, then

$$\left| \mathbb{P}_\pi(T_A > t) - \exp\left(-\frac{t}{\mathbb{E}_\pi(T_A)}\right) \right| \leq \frac{t_{\mathrm{rel}}}{\mathbb{E}_\pi(T_A)},$$

and

$$\frac{1}{\mathbb{E}_\pi(T_A)}\left(1 - \frac{2t_{\mathrm{rel}} + t}{\mathbb{E}_\pi(T_A)}\right) \leq f_{T_A}(t) \leq \frac{1}{\mathbb{E}_\pi(T_A)}\left(1 + \frac{t_{\mathrm{rel}}}{2t}\right).$$

Consider the product chain and take A to be the diagonal set. We have $E_\pi(T_A) = t_{\mathrm{meet}}$.

Beyond $\mathbb{Z}^d$-Mean Field Predictions

[Oliveira, 2013] proved the Aldous-Fill conjecture under the condition $t_{\mathrm{mix}} \ll t_{\mathrm{meet}}$ (which is proved by Hermon to be equivalent to $t_{\mathrm{rel}} \ll t_{\mathrm{meet}}$).
The time it takes to make $n-h$ collisions is about

$$t_{\mathrm{meet}} \sum_{i \geq h} \frac{e_i}{i(i-1)/2} \sim \frac{2t_{\mathrm{meet}}}{h}.$$

$$\frac{2t_{\mathrm{meet}}}{h} = t \Rightarrow h = \frac{2t_{\mathrm{meet}}}{t}.$$

Hence the number of particles that remain at t is roughly h and we have the prediction

$$P_t = \frac{h}{n} \sim \frac{2t_{\mathrm{meet}}}{nt}.$$

Equivalence of the two predictions

Two predictions for P_t (for t large)

$$P_t \sim \frac{1}{t\alpha_t}$$

where $\alpha_t = \mathbb{P}_{o,\nu_o}(\tau_{\text{meet}} > t)$ (ν_o is a random neighbor of o),

$$P_t \sim \frac{2t_{\text{meet}}}{n} \text{ for finite graphs}$$

are equivalent to each other for many graphs satisfying certain transience conditions (e.g., α_t is almost a positive constant for large t).
Kac's formula!

Main Results: finite graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

Two predictions holds as long as $1 \ll t \ll t_{coal}$ *(called the Big Bang regime since the number of particles is evolving rapidly in this regime) for*

- *transitive graphs (transitivity means the graph looks the same from every vertex)* G_n *such that* $diam(G_n)^2 \ll n/\log n$,
- *Configuration Model* $\mathbb{CM}(n, D)$ *with* $3 \leq D < M$.
  If D *is a constant* d *then* $\mathbb{CM}(n, D)$ *is random* d*-regular graph.*

Configuration model

Construction of the configuration model $\mathbb{CM}_n(D)$

- Let D be a probability measure on $\mathbb{Z}_+$, and $n \in \mathbb{Z}_+$.
- We take n vertices labeled $1, \ldots, n$, and $d_1, \ldots, d_n$ i.i.d. sampled from D.
- For each vertex i we attach d_i half edges to it. Then we get G_n by uniformly matching all half edges, conditioned on $\sum_{i=1}^n d_i$ being even.

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The *local weak limit* $\mathbb{UGT}(D)$ of $\mathbb{CM}_n(D)$ is a unimodular Galton-Watson tree where

- the root has offspring distribution D
- later generations have offspring distribution D^*:

$$\mathbb{P}(D^* = k) := \frac{(k+1)\mathbb{P}(D = k+1)}{\sum_{i=0}^{\infty} i\mathbb{P}(D = i)}$$

Main Results: infinite Graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

The first prediction $P_t \sim 1/(t\alpha)$ *as* $t \to \infty$ *where*

$$\alpha = \mathbb{P}_{o,\nu_o}(\tau_{meet} = \infty).$$

holds for

- *all transient transitive unimodular graphs (including all Cayley graphs and all amenable graphs(=graphs with subexponential decay of return probability)),*
- *unimodular Galton-Watson tree UGT(D). If D is a constant d then* $UGT(D) = \mathbb{T}^d$.

Ingredients of the proof

Using the machinery in the proof of $\mathbb{Z}^d$ case by Braomson-Griffeath, it suffices to

- give an upper bound of P_t that differs from the 'true value' of P_t by a multiplicative constant,
- show that the coalescence probability

$$\mathbb{P}_{\pi^k}(\mathbb{C}(X_1, \ldots, X_{k+1}) \leq t) \sim (k+1)! \left(\frac{t}{t_{\mathrm{meet}}} \right)^k .$$

Another indication of mean field! B-G proof heavily relies on the specific geometric structure of $\mathbb{Z}^d$.

Solution

- For the first part, we show that

$$c \frac{\inf_x \int_0^t p_s(x,x)ds}{t} \leq P_t \leq C \frac{\sup_x \int_0^t p_s(x,x)ds}{t}.$$

- For the second part, we use the reversibility of random walk to transform collision probability to non-colliding probability. If two forward paths collide at t then (after reversing time) the backward paths don't collide in $[0, t]$.

Open Question

Our results are stated for the expectation of the number of occupied sites (which can be upgraded to weak law of large numbers using negative correlation). What about fluctuations?

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Thanks!